

BEILINSON-BERNSTEIN LOCALIZATION OVER THE HARISH-CHANDRA CENTER

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ABSTRACT. We present a simple proof of a strengthening of the derived Beilinson-Bernstein localization theorem using the formalism of descent in derived algebraic geometry. The arguments and results apply to arbitrary modules without the need to fix infinitesimal character. Roughly speaking, we demonstrate that all $\mathfrak{U}\mathfrak{g}$ -modules are the invariants, or equivalently coinvariants, of the action of intertwining functors (a refined form of Weyl group symmetry). This is a quantum version of descent for the Grothendieck-Springer simultaneous resolution.

CONTENTS

1. Introduction	1
1.1. Beilinson-Bernstein via Barr-Beck	3
1.2. Beilinson-Bernstein via Hecke symmetry	3
1.3. Comparison with K -theory	6
1.4. Classical limit	6
1.5. Categorical context	8
1.6. Acknowledgements	8
2. Grothendieck-Springer resolution	8
2.1. Recollections	8
2.2. Descent pattern	10
2.3. Specified eigenvalues	10
3. Beilinson-Bernstein localization	11
3.1. Quantization	12
3.2. Universal Weyl sheaf	14
3.3. Specified infinitesimal character	16
3.4. Example: SL_2	17
References	18

1. INTRODUCTION

The Beilinson-Bernstein localization theorem is the quintessential result in geometric representation theory. To recall its statement, fix a complex reductive group G , Borel subgroup $B \subset G$ with unipotent radical $N \subset B$, and let $H = B/N$ denote the universal Cartan. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}$ and \mathfrak{h} denote their respective Lie algebras, and W the Weyl group of G .

Fix $\lambda \in \mathfrak{h}^*$, and let $\mathcal{P}_\lambda = \mathcal{D}_\lambda(G/B)$ denote the dg category of λ -twisted \mathcal{D} -modules on the flag variety.¹ By λ -twisted \mathcal{D} -modules, one can take \mathcal{D} -modules on the basic affine space G/N which

¹Though one can phrase everything we discuss in more traditional language, it is convenient and often enlightening to work in the general setting of ∞ -categories and derived algebraic geometry. The term “dg category” will stand throughout for a \mathbf{C} -linear pre-triangulated dg category, or equivalently \mathbf{C} -linear stable ∞ -category, and all operations with dg categories will be taken in the derived sense, i.e., we work in the ∞ -category of dg categories. See Section 1.5 below for a brief summary of our working context.

are weakly H -equivariant and have monodromy λ along the H -orbits. (The notation \mathcal{P}_λ reflects the analogy with principal series representations of real or p -adic groups.)

Let $\mathfrak{U}\mathfrak{g}$ denote the universal enveloping algebra of \mathfrak{g} , and $\mathfrak{Z}\mathfrak{g} \simeq \mathcal{O}(\mathfrak{h}^*/W) \subset \mathfrak{U}\mathfrak{g}$ the Harish Chandra center. Let $\mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda]}$ denote the dg category of $\mathfrak{U}\mathfrak{g}$ -modules with infinitesimal character the W -orbit $[\lambda] \in \mathfrak{h}^*/W \simeq \text{Spec } \mathfrak{Z}\mathfrak{g}$.

Theorem 1.1 ([BB]). *For $\lambda \in \mathfrak{h}^*$ regular, localization and global sections provide inverse equivalences of derived categories*

$$\Delta : \mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda]} \xleftarrow{\sim} \mathcal{P}_\lambda : \Gamma$$

Remark 1.2 (Refinements). There are several standard refinements of the above statement.

1) One can bound the cohomological amplitude of Γ by the length of the Weyl group element $w \in W$ for which $w \cdot \lambda$ is dominant. In particular, when λ is dominant, the equivalence preserves the corresponding abelian categories.

2) For $\lambda \in \mathfrak{h}^*$ singular, one can exhibit $\mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda]}$ as the quotient of \mathcal{P}_λ by the kernel of Γ . Alternatively, one can realize $\mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda]}$ as the subcategory of \mathcal{P}_λ left-orthogonal to the kernel of Γ . These discrepancies can be bridged via localization on partial flag varieties [K, BMR1, BK].

3) The theorem respects the natural G -actions and so identifies equivariant objects on each side. Thus for a subgroup $K \subset G$, it provides an equivalence between Harish Chandra $(\mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda]}, K)$ -modules and K -equivariant λ -twisted \mathcal{D} -modules.

4) One can naturally extend the theorem to generalized monodromy and infinitesimal character and thus obtain an equivalence not for fixed $\lambda \in \mathfrak{h}^*$ but over its formal neighborhood λ^\wedge within \mathfrak{h}^* . Namely, let $\mathcal{P}_{\lambda^\wedge} = \mathcal{D}_{\lambda^\wedge}(G/B)$ denote the dg category of weakly H -equivariant \mathcal{D} -modules on G/N with monodromy in λ^\wedge . Similarly, let $\mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda^\wedge]}$ denote the dg category of $\mathfrak{U}\mathfrak{g}$ -modules with infinitesimal character in the W -orbit $[\lambda^\wedge]$ or equivalently formal neighborhood $[\lambda]^\wedge$. Then when λ is regular, localization and global sections provide inverse equivalences

$$\Delta : \mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda^\wedge]} \xleftarrow{\sim} \mathcal{P}_{\lambda^\wedge} : \Gamma$$

Alternatively, one can equip the original statement of Theorem 1.1 with the symmetries that control the extension to the formal neighborhood.

Despite its dramatic importance, which is hard to overestimate, there nevertheless remain several deficiencies of Theorem 1.1, even after taking into account the above refinements.

First, and foremost of our motivations, the theorem applies to modules with fixed, or at most generalized, infinitesimal character. This makes it difficult to apply to questions of harmonic analysis, which typically involve the geometry or topology of families of representations (for example, the Plancherel formula and Baum-Connes conjecture).

Second, it is unnatural and thus often confusing that the parameters on the two sides of the theorem do not match: to localize modules with infinitesimal character $[\lambda] \in \mathfrak{h}^*/W$, we must choose a representative lift $\lambda \in \mathfrak{h}^*$. It is worth noting the impact of this choice: the geometry of the corresponding \mathcal{D} -modules – most basically, the dimension of their support – depends on this choice.

Third, the theorem does not apply as stated to singular infinitesimal character where the category of \mathcal{D} -modules is larger than the corresponding category of modules. As mentioned above, one can bridge this discrepancy via localization on partial flag varieties. But this seems to only increase the distance from a uniform statement over all infinitesimal characters.

In this paper, we present a natural refinement of the derived Beilinson-Bernstein localization theorem which simultaneously corrects these three drawbacks by invoking descent. A naive paraphrase of the main result asserts: “the category of $\mathfrak{U}\mathfrak{g}$ -modules is equivalent to the Weyl group invariants in the category of all monodromic \mathcal{D} -modules on G/N ”.

In what immediately follows, we first state a precise version of this result, then explain the paraphrase and a “classical limit” for quasicohherent sheaves on the Grothendieck-Springer resolution.

1.1. Beilinson-Bernstein via Barr-Beck. A natural setting for simultaneously localizing over all infinitesimal characters is the dg category $\mathcal{P} = \mathcal{D}_{\text{mon}}(G/N)$ of all weakly H -equivariant \mathcal{D} -modules on the basic affine space G/N (see for example [B]). The subscript “*mon*” stands for the term monodromic, which we will understand to be a synonym for weakly H -equivariant, and in particular not imply specified monodromy along the H -orbits. (The notation \mathcal{P} reflects the analogy with the universal principal series representation of a real group or the universal unramified principal series representation of a p -adic group.)

The natural right H -action on G/N induces an identification of $\mathfrak{U}\mathfrak{h} \simeq \mathcal{O}(\mathfrak{h}^*)$ with central H -invariant differential operators on G/N . This equips \mathcal{P} with a linear structure over the base \mathfrak{h}^* with fiber at $\lambda \in \mathfrak{h}^*$ the previously encountered dg category $\mathcal{P}_\lambda = \mathcal{D}_\lambda(G/B)$ of λ -twisted \mathcal{D} -modules. Similarly, restricting to the formal neighborhood λ^\wedge , we recover the dg category $\mathcal{P}_{\lambda^\wedge} = \mathcal{D}_{\lambda^\wedge}(G/B)$ of weakly H -equivariant \mathcal{D} -modules with monodromy in λ^\wedge .

The natural left G -action on G/N induces an embedding of $\mathfrak{U}\mathfrak{g}$ in H -invariant differential operators on G/N . This gives rise to an adjunction

$$\Delta : \mathfrak{U}\mathfrak{g}\text{-mod} \longleftrightarrow \mathcal{P} : \Gamma$$

between the dg category of all $\mathfrak{U}\mathfrak{g}$ -modules and \mathcal{P} . Note that here Γ denotes the H -invariants in the usual global sections functor. The functors intertwine the linear structure over \mathfrak{h}^*/W on the left and that over \mathfrak{h}^* on the right.

An important observation is that the adjunction is naturally *ambidextrous*: Δ is canonically the right adjoint of Γ as well. This is a reflection of the Calabi-Yau structure of \mathcal{P} , the key ingredient in the approach of [BMR1, BMR2] to establishing Beilinson-Bernstein equivalences (following an argument of [BKR] in the setting of the McKay correspondence, and extended to localization for quantum symplectic resolutions in [MN]).

We will refer to the composition

$$\mathcal{W} = \Delta \circ \Gamma \in \text{End}(\mathcal{P})$$

as the *Weyl functor*. It naturally comes equipped with the structure of a monad and a comonad, or in other words, an algebra and coalgebra object in endofunctors of \mathcal{P} . In fact, one could organize all of the structure in what could rightfully be called a *Frobenius monad*. We will use this term as evocative shorthand for the monad of an ambidextrous adjunction (and will not attempt to independently formalize it in the ∞ -categorical setting; see Section 1.5 below for further discussion and references to the discrete setting).

Another crucial feature of Δ is its conservativity: no objects localize to zero (an easy consequence of the general localization formalism). Now the Barr-Beck theorem, in its ∞ -categorical form due to Lurie, allows us to describe $\mathfrak{U}\mathfrak{g}\text{-mod}$ in terms of \mathcal{P} and the Weyl functor \mathcal{W} with its natural structures. Let us write $\mathcal{P}_{\mathcal{W}}$ and $\mathcal{P}^{\mathcal{W}}$ for the respective dg categories of \mathcal{W} -modules and \mathcal{W} -comodules in \mathcal{P} .

Theorem 1.3 (Barr-Beck version of localization). *There are canonical equivalences*

$$\mathcal{P}_{\mathcal{W}} \simeq \mathfrak{U}\mathfrak{g}\text{-mod} \simeq \mathcal{P}^{\mathcal{W}}$$

between the dg categories of \mathcal{W} -modules and \mathcal{W} -comodules in \mathcal{P} and the category of $\mathfrak{U}\mathfrak{g}$ -modules.

Remark 1.4. For fixed regular $[\lambda] \in \mathfrak{h}^*/W$, the arguments of [BMR1] utilize the ambidextrous property and the indecomposability of $\mathcal{D}_\lambda(G/B)$ to deduce the Beilinson-Bernstein equivalence. For singular parameters or families of infinitesimal characters, the indecomposability fails, but we recover $\mathfrak{U}\mathfrak{g}\text{-mod}$ as a summand of \mathcal{P} in the form of an isotypic component $\mathfrak{U}\mathfrak{g}\text{-mod} = \mathcal{P}^{\mathcal{W}}$.

1.2. Beilinson-Bernstein via Hecke symmetry. In order to exploit the formal assertion of Theorem 1.3, we need to identify the Weyl functor \mathcal{W} concretely in terms of the symmetries of \mathcal{P} . By construction, the category \mathcal{P} carries two fundamental commuting actions.

On the one hand, the left G -action on G/N naturally equips \mathcal{P} with the structure of *smooth G -category*. By this, we mean an algebraic G -action that is infinitesimally trivialized, or in other

words, the induced action of the formal group of G is trivialized. This can be formalized by saying that \mathcal{P} is a module for the monoidal dg category $\mathcal{D}(G)$ of \mathcal{D} -modules on G under convolution.

On the other hand, the Hecke category $\mathcal{H} = \mathcal{D}_{\text{bimon}}(N \backslash G / N)$ of bimonodromic \mathcal{D} -modules on $N \backslash G / N$ naturally acts on \mathcal{P} on the right by intertwining functors. In the same way that \mathcal{P} is linear over the base \mathfrak{h}^* , the Hecke category \mathcal{H} is linear over the base $\mathfrak{h}^* \times \mathfrak{h}^*$, and its monoidal structure is compatible with convolution of bimodules over $\mathfrak{h}^* \times \mathfrak{h}^*$.

One can view the Hecke category \mathcal{H} as the monodromic generalization of the familiar finite Hecke category $\mathcal{D}(B \backslash G / B)$, which in turn is a categorical form of the finite Hecke algebra or Artin braid group. More generally, one finds the Hecke category $\mathcal{H}_{\lambda, \mu} = \mathcal{D}_{\lambda, \mu}(B \backslash G / B)$ of λ, μ -twisted \mathcal{D} -modules as the fiber of \mathcal{H} at a point $(\lambda, \mu) \in \mathfrak{h}^* \times \mathfrak{h}^*$. Similarly, restricting to the formal neighborhood $\lambda^\wedge \times \mu^\wedge$, one finds the Hecke category $\mathcal{H}_{\lambda^\wedge, \mu^\wedge} = \mathcal{D}_{\lambda^\wedge, \mu^\wedge}(B \backslash G / B)$ of weakly $H \times H$ -equivariant \mathcal{D} -modules with monodromy in $\lambda^\wedge \times \mu^\wedge$.

Remark 1.5. At first glance, it might look easier to work with strictly λ -twisted \mathcal{D} -modules rather than generalized twisted \mathcal{D} -modules with monodromy in λ^\wedge . But since λ^\wedge is flat over the base \mathfrak{h}^* unlike the point λ itself, tensor products and ultimately convolution patterns restrict to λ^\wedge in a less intricate way. Alternatively, one can equip strictly λ -twisted \mathcal{D} -modules with the symmetries that control their extension to λ^\wedge , but keeping track of this extra Koszul dual structure can be less intuitive than simply working over λ^\wedge .

Remark 1.6. Here are some simple observations to help orient the reader.

The restriction $\mathcal{H}_{\lambda^\wedge, \mu^\wedge}$ vanishes unless $\mu = w \cdot \lambda$, for some Weyl group element $w \in W$. The restrictions $\mathcal{H}_{\lambda^\wedge, \mu^\wedge}$ and $\mathcal{H}_{\lambda'^\wedge, \mu'^\wedge}$ are non-canonically equivalent if there are Weyl group elements $w, w' \in W$ such that $\lambda' = w \cdot \lambda, \mu' = w' \cdot \mu$. The restrictions $\mathcal{H}_{\lambda^\wedge, \mu^\wedge}$ and $\mathcal{H}_{\lambda'^\wedge, \mu'^\wedge}$ are canonically equivalent if the difference of parameters is integral $\lambda - \lambda', \mu - \mu' \in \Lambda^* \subset \mathfrak{h}^*$.

The monoidal structure of \mathcal{H} descends to compatible compositions

$$\mathcal{H}_{\lambda^\wedge, \mu^\wedge} \otimes \mathcal{H}_{\mu^\wedge, \nu^\wedge} \longrightarrow \mathcal{H}_{\lambda^\wedge, \nu^\wedge}$$

Moreover, the restriction maps $\mathcal{H} \rightarrow \mathcal{H}_{\lambda^\wedge, \mu^\wedge}$ are compatible with the above restricted compositions. In particular, the diagonal restriction map $\mathcal{H} \rightarrow \mathcal{H}_{\lambda^\wedge, \lambda^\wedge}$ is a monoidal map of monoidal categories.

Remark 1.7. A result of [BN] asserts that for any fixed $\lambda \in \mathfrak{h}^*$, the diagonally λ -twisted Hecke categories $\mathcal{H}_{\lambda, \lambda} = \mathcal{D}_{\lambda, \lambda}(B \backslash G / B)$, $\mathcal{H}_{\lambda^\wedge, \lambda^\wedge} = \mathcal{D}_{\lambda^\wedge, \lambda^\wedge}(B \backslash G / B)$ are both categorified analogues of finite dimensional semisimple Frobenius algebras: they are the values on a point of extended oriented two-dimensional topological field theories. More precisely, they are two-dualizable Calabi-Yau algebra objects in dg categories.

Along the way, as a simple application of results of [BN], we will establish the following basic relationship between the above commuting symmetries.

Theorem 1.8. *There is a monoidal equivalence*

$$\mathcal{H} \simeq \text{End}_{\mathcal{D}(G)}(\mathcal{P})$$

between the Hecke category \mathcal{H} and $\mathcal{D}(G)$ -linear endofunctors of \mathcal{P} .

Remark 1.9. Specializing over subsets of the base \mathfrak{h}^* immediately provides analogous assertions for monodromic \mathcal{D} -modules with prescribed monodromies.

Observe that the adjoint G -action on \mathfrak{g} naturally equips $\mathfrak{U}\mathfrak{g}\text{-mod}$ with the structure of smooth G -category. As was emphasized in [BD] (see also [FG]), the localization and global sections functors commute with the corresponding smooth G -actions. Therefore, or as can be verified independently, the Weyl functor \mathcal{W} must be represented by an object of the Hecke category \mathcal{H} .

Definition 1.10. The *universal Weyl sheaf* $\mathcal{W} \in \mathcal{H}$ is the sheaf of differential operators on $N \backslash G / N$ with its canonical weakly H -biequivariant structure.

Theorem 1.11 (Hecke version of localization). *The universal Weyl sheaf \mathcal{W} is naturally a algebra and coalgebra in \mathcal{H} . There are canonical equivalences*

$$\mathcal{P}_{\mathcal{W}} \simeq \mathfrak{U}\mathfrak{g}\text{-mod} \simeq \mathcal{P}^{\mathcal{W}}$$

between the categories of \mathcal{W} -modules and \mathcal{W} -comodules in \mathcal{P} and the category of $\mathfrak{U}\mathfrak{g}$ -modules.

Altogether, one can view the universal Weyl sheaf \mathcal{W} as a categorical idempotent, providing a lift of the idempotent in the groupoid algebra governing descent along the quotient map $\mathfrak{h}^* \rightarrow \mathfrak{h}^*//W$. It is clarifying to reverse our momentum and regard it as a family of idempotents as we vary the infinitesimal character. We include here a brief discussion of the two most extreme cases.

Remark 1.12. It is worth pointing out first of all that while $\mathcal{W} \in \mathcal{H}$ is the sheaf of differential operators on $N \backslash G/N$ so “large”, its fibers $\mathcal{W}_{\lambda, \mu} \in \mathcal{H}_{\lambda, \mu}$ are regular holonomic twisted \mathcal{D} -modules on $B \backslash G/B$ so “small”. This is a twisted instance of the general fact that on a quotient stack $H \backslash X$ such that X is smooth and H is affine algebraic acting with finitely many orbits in X , the sheaf of differential operators on $H \backslash X$ is regular holonomic. In fact, any coherent \mathcal{D} -module on $H \backslash X$ is regular holonomic.

At one extreme, when $\lambda \in \mathfrak{h}^*$ is regular, consider the direct sum of the potential targets for localization

$$\mathcal{P}_{[\lambda^\wedge]} = \bigoplus_{w \in W} \mathcal{P}_{w \cdot \lambda^\wedge}$$

and the corresponding Hecke category

$$\mathcal{H}_{[\lambda^\wedge]} = \bigoplus_{w, w' \in W} \mathcal{H}_{w \cdot \lambda^\wedge, w' \cdot \lambda^\wedge}$$

First, for $\lambda \in \mathfrak{h}^*$ generic (with respect to the notion of integrality coming from coroot functionals), we have that

$$\mathcal{H}_{w \cdot \lambda^\wedge, w' \cdot \lambda^\wedge} \simeq \mathcal{Q}(\lambda^\wedge), \quad \text{for all } w, w' \in W,$$

where $\mathcal{Q}(\lambda^\wedge)$ denotes the dg category of quasicohherent sheaves on the formal neighborhood of $\lambda \in \mathfrak{h}^*$ or equivalently that of $[\lambda] \in \mathfrak{h}^*//W$. The monoidal structure of $\mathcal{H}_{[\lambda^\wedge]}$ is simply that of the groupoid algebra of the regular Weyl groupoid

$$W \times W \rightrightarrows W$$

with scalars in the constant tensor category $\mathcal{Q}(\lambda^\wedge)$. Furthermore, the restriction $\mathcal{W}_{[\lambda^\wedge]} \in \mathcal{H}_{[\lambda^\wedge]}$ of the universal Weyl sheaf corresponds to the direct sum of the structure sheaf $\mathcal{O}_{\lambda^\wedge} \in \mathcal{Q}(\lambda^\wedge)$ in each factor. Thus it provides the categorical analogue of the constant idempotent in the groupoid algebra of W . As expected, its modules and comodules in $\mathcal{P}_{[\lambda^\wedge]}$ are equivalent to a single copy of $\mathcal{P}_{\lambda^\wedge}$.

Leaving behind the generic case, for $\lambda \in \mathfrak{h}^*$ regular integral, the Hecke category $\mathcal{H}_{[\lambda^\wedge]}$ now contains all of the combinatorics of Kazhdan-Lusztig theory. A common approach to capture this structure is to focus on the exceptional collections of standard or costandard objects associated to Schubert cells. Their convolutions provide dual realizations of the groupoid algebra of the natural Artin braid group action on the Weyl group

$$B_W \times W \rightrightarrows W$$

with scalars in the constant tensor category $\mathcal{Q}(\lambda^\wedge)$. In fact, one can embed the regular Weyl groupoid inside the above braid groupoid by choosing appropriate lifts of minimal standard or costandard objects depending on whether multiplication increases or decreases length. With this observation in hand, the restriction $\mathcal{W}_{[\lambda^\wedge]} \in \mathcal{H}_{[\lambda^\wedge]}$ corresponds to a direct sum of specified standard and costandard objects giving the constant idempotent of W . Thus as expected, its modules and comodules in $\mathcal{P}_{[\lambda^\wedge]}$ are equivalent to $\mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda^\wedge]}$ in the form of a single copy of $\mathcal{P}_{\lambda^\wedge}$.

Remark 1.13. The above picture for $\lambda \in \mathfrak{h}^*$ regular integral gives a nice framework for understanding the Borel-Weil-Bott theorem. Let us focus on the objects of $\mathcal{P}_{[\lambda^\wedge]}$ given by the shifted line bundles $\mathcal{O}(w \cdot \lambda)[\ell(w)] \rightarrow G/B$, where $\ell(w)$ denotes the length of $w \in W$. Their global sections are the irreducible G -representation $V_{[\lambda]}$ placed in degree zero thanks to their original shifts by length.

The summands of the idempotent $\mathcal{W}_{[\lambda^\wedge]} \in \mathcal{H}_{[\lambda^\wedge]}$ intertwine the shifted line bundles, increasing or decreasing degree depending on what happens to length.

Finally, as $\lambda \in \mathfrak{h}^*$ specializes to become singular, the geometry becomes more interesting, ultimately reflecting the nontrivial scheme structure of the quotient map $\mathfrak{h}^* \rightarrow \mathfrak{h}^*/W$ along its ramification locus. In the most singular case $\lambda = 0 \in \mathfrak{h}^*$, recall that the target of localization $\mathcal{P}_{[0^\wedge]} \simeq \mathcal{P}_{0^\wedge}$ is dramatically different from the singular category $\mathfrak{Ug}\text{-mod}_{[0^\wedge]}$. The Hecke category $\mathcal{H}_{[0^\wedge]} = \mathcal{H}_{0^\wedge, 0^\wedge}$ controlling this difference can be viewed in dual standard and costandard ways as a categorical analogue of the Artin braid group algebra (rather than groupoid algebra as above).

To concretely discuss the restriction $\mathcal{W}_{[0^\wedge]} \in \mathcal{H}_{[0^\wedge]}$, let us focus on its convolution against the object $\mathcal{O}(-\rho) \in \mathcal{P}_{[0^\wedge]}$ given by the canonical bundle of G/B . One can calculate that

$$\mathcal{T} = \mathcal{W}_{[0^\wedge]} * \mathcal{O}(-\rho)$$

is the tilting sheaf given by the projective cover of the skyscraper at the closed Schubert cell. It is well known that \mathcal{T} governs the singular category (as in the work of Soergel [S]); for example, the kernel of the global sections functor to $\mathfrak{Ug}\text{-mod}_{[0^\wedge]}$ is its right-orthogonal.

1.3. Comparison with K -theory. We would like to interpret Theorem 1.11 as a refinement of results in K -theory which are closer in form to the naive paraphrase: “the category of \mathfrak{Ug} -modules is equivalent to the Weyl group invariants in the category of all monodromic \mathcal{D} -modules on G/N ”.

First, let us recall K -theory versions of the Weyl character formula and Borel-Weil-Bott theorem from [BH] (where the context is equivariant KK -theory and the results are closely related to the Baum-Connes conjecture for Lie groups). To proceed in this setting, let $G_c \subset G$ be a maximal compact subgroup, and $T_c \subset G_c$ a maximal torus, so that we have $G/B \simeq G_c/T_c$. Natural morphisms relate the representation ring of G_c and the equivariant K -theory of the flag manifold

$$\Delta : K_{G_c}^*(pt) \longleftrightarrow K_{G_c}^*(G_c/T_c) : \Gamma$$

where Γ is the equivariant index (Borel-Weil-Bott construction) and Δ is given by pullback followed by multiplication by the virtual bundle Ω . Note that for a representation $V \in K_{G_c}^*(pt)$, the virtual bundle $\Delta(V)$ can be identified with the complex that fiberwise computes \mathfrak{n} -homology (where \mathfrak{n} is the unipotent radical of the stabilizer of a point in G/B). In other words, the virtual bundle $\Delta(V)$ is the K -theory image of the Beilinson-Bernstein localization of V .

The Weyl group W acts (non-holomorphically) from the right on G_c/T_c , and the main theorem of [BH] calculates (in the context of equivariant KK -theory) that $\Gamma \circ \Delta = |W| \text{Id}$, and when restricted to the W -invariants, $\Delta \circ \Gamma = |W| \text{Id}$. Identifying the virtual character of Ω with the Weyl denominator, one recovers the Weyl character formula. In fact, the main theorem of [BH] calculates that $\Delta \circ \Gamma$ is given by the standard idempotent in the group algebra of W (the projector to the trivial representation), realized as a sum of standard intertwining operators. Note that the group algebra $\mathbb{C}W$ is a Frobenius algebra, and so W -invariants and coinvariants are both identified with the summand given by the image of the standard idempotent.

In our present categorical setting, the action of the Weyl group on equivariant K -theory is replaced by the action of the Hecke category $\mathcal{H} = \mathcal{D}_{bimon}(N \backslash G/N)$ on the category $\mathcal{P} = \mathcal{D}_{mon}(G/N)$. For fixed integral $\lambda \in \mathfrak{h}^*$, the corresponding Hecke category $\mathcal{H}_{\lambda, \lambda} = \mathcal{D}_{\lambda, \lambda}(B \backslash G/B)$ of Kazhdan-Lusztig theory has Grothendieck group the group algebra $\mathbb{C}W$. But in fact standard bases of $\mathcal{H}_{\lambda, \lambda}$ provide actions on categories not of the Weyl group W but of the corresponding Artin braid group. The Frobenius monad \mathcal{W} is the categorified analogue of the standard idempotent, with \mathcal{W} -modules playing the role of W -invariants and \mathcal{W} -comodules that of W -coinvariants.

1.4. Classical limit. The preceding description of Beilinson-Bernstein localization has a natural classical analogue for quasicohherent sheaves on the Springer resolution. Its interpretation as a form of proper descent becomes evident in this setting. We will proceed in the context of derived algebraic geometry, and in particular that of perfect stacks as introduced in [BFN]. In particular,

let $\pi : X \rightarrow Y$ denote any morphism of perfect stacks, and $X \times_Y X$ the corresponding derived fiber product.

First, consider the symmetric monoidal dg category $\mathcal{Q}(Y)$ of quasicoherent sheaves equipped with tensor product, and its natural module dg category $\mathcal{Q}(X)$ under the pullback π^* . The dg category $\mathcal{Q}(X \times_Y X)$ is monoidal with respect to convolution, acts on $\mathcal{Q}(X)$ by endofunctors with a natural $\mathcal{Q}(Y)$ -linear structure, thus leading to a monoidal equivalence

$$\mathcal{Q}(X \times_Y X) \xrightarrow{\sim} \text{End}_{\mathcal{Q}(Y)}(\mathcal{Q}(X)).$$

The adjunction (π^*, π_*) on quasicoherent sheaves associated to $\pi : X \rightarrow Y$ defines a comonad $T^\vee = \pi^* \pi_*$ acting on $\mathcal{Q}(X)$, and it is easy to see that T^\vee is represented by the coalgebra object

$$\mathcal{A} = \mathcal{O}_{X \times_Y X} \in \mathcal{Q}(X \times_Y X).$$

Observe that \mathcal{A} is simply the groupoid coalgebra (functions on the groupoid with convolution coproduct) for the descent groupoid $X \times_Y X$ acting on X , and an \mathcal{A} -comodule structure simply enforces invariance along the fibers of π . When π is faithfully flat, then descent holds by [L4, 7], providing an equivalence

$$\mathcal{Q}(Y) \simeq \mathcal{Q}(X)^{\mathcal{A}}.$$

Such flatness will not hold in our setting, but we will know that π^* is conservative and cocontinuous and thus descent holds by the Barr-Beck-Lurie theorem.

Next, let us assume that π is proper and surjective on field points. Consider the symmetric monoidal dg category $\mathcal{Q}^!(Y)$ of ind-coherent sheaves equipped with $!$ -tensor product, and its natural module dg category $\mathcal{Q}^!(X)$ under the pullback $\pi^!$. The dg category $\mathcal{Q}^!(X \times_Y X)$ is monoidal with respect to convolution, acts on $\mathcal{Q}^!(X)$ by endofunctors with a natural $\mathcal{Q}^!(Y)$ -linear structure, thus leading to a monoidal functor which is typically not an equivalence

$$\mathcal{Q}^!(X \times_Y X) \longrightarrow \text{End}_{\mathcal{Q}^!(Y)}(\mathcal{Q}^!(X)).$$

The adjunction $(\pi_*, \pi^!)$ on ind-coherent sheaves associated to $\pi : X \rightarrow Y$ defines a monad $T = \pi^! \pi_*$ acting on $\mathcal{Q}^!(X)$, and it is easy to see that T is represented by an algebra object

$$\mathcal{A}^! = \omega_{X \times_Y X/X} \in \mathcal{Q}^!(X \times_Y X).$$

Observe that $\mathcal{A}^!$ is simply the groupoid algebra (relative volume forms on the groupoid with convolution product) for the descent groupoid $X \times_Y X$ acting on X , and an $\mathcal{A}^!$ -module structure simply realizes coinvariance along the fibers of π . The proper descent theorem of [P, Proposition A.2.8] and [G2, 7.2.2] provides an equivalence

$$\mathcal{Q}^!(Y) \simeq \mathcal{Q}^!(X)_{\mathcal{A}^!}.$$

Finally, let us now assume that X and Y are both smooth, so that we have canonical identifications $\mathcal{Q}(X) \simeq \mathcal{Q}^!(X)$ and $\mathcal{Q}(Y) \simeq \mathcal{Q}^!(Y)$. Moreover, let us assume that π is proper, surjective on field points, and *crepant*, or in other words, Calabi-Yau of dimension zero in that we are given a trivialization of its relative dualizing sheaf and thus an identification $\pi^* \simeq \pi^!$. In particular, this implies that $\pi^* \simeq \pi^!$ are simultaneously continuous and cocontinuous, as well as conservative. Then $T \simeq T^\vee$ is a Frobenius monad, the groupoid algebra $\omega_{X \times_Y X/X} \simeq \mathcal{O}_{X \times_Y X}$ is a Frobenius algebra object, and there are equivalences

$$\mathcal{Q}(X)^T \simeq \mathcal{Q}(Y) \simeq \mathcal{Q}(X)_T.$$

The prime example for these restrictive hypotheses is the Grothendieck-Springer simultaneous resolution $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ with descent groupoid the Grothendieck-Steinberg variety. The resulting descent picture is precisely the classical limit of Beilinson-Bernstein localization. We develop the details of this in Section 2, in particular its various specializations over different regions in the adjoint quotient.

1.5. Categorical context. We will work throughout in the language of derived algebraic geometry following [L1, L2]; we refer the reader to [BFN, BN] for some gentle discussion of this context and its basic tools. We will work throughout over the complex numbers \mathbb{C} .

The words “category” and “dg category” will stand for either a \mathbb{C} -linear pre-triangulated dg category or a \mathbb{C} -linear stable ∞ -category, and we refer to [G2] for a general homotopical treatment of dg categories. Such categories fit into two related contexts: 1) $St_{\mathbb{C}}$ the symmetric monoidal ∞ -category of stable presentable \mathbb{C} -linear ∞ -categories with morphisms continuous functors, and 2) $st_{\mathbb{C}}$ the symmetric monoidal ∞ -category of small stable idempotent-complete \mathbb{C} -linear ∞ -categories with morphisms exact functors. We say that a functor between stable ∞ -categories is *continuous* if it preserves coproducts, *exact* if it preserves zero objects and finite colimits, and *proper*² if it preserves compact objects.

Taking ind-objects defines a faithful symmetric monoidal functor $\text{Ind} : st_{\mathbb{C}} \rightarrow St_{\mathbb{C}}$. It admits a left inverse on the subcategory of proper functors given by passing to compact objects. Any category $\mathcal{C} \in st_{\mathbb{C}}$ is dualizable with dual the opposite category $\mathcal{C}^{op} \in st_{\mathbb{C}}$. Thus any category $\text{Ind } \mathcal{C} \in St_{\mathbb{C}}$ is dualizable with dual the restricted opposite category $(\text{Ind } \mathcal{C})^{\vee} = \text{Ind}(\mathcal{C}^{op}) \in Stab_{\mathbb{C}}$.

We will make heavy use of the theory of adjunctions, monads and comonads, and the Barr-Beck-Lurie theorem [L2]. Given a monad T or comonad T^{\vee} on a category \mathcal{C} , we denote by

$$\mathcal{C}_T = \text{Mod}_T(\mathcal{C}) \quad \mathcal{C}^{T^{\vee}} = \text{Comod}_{T^{\vee}}(\mathcal{C})$$

the respective category of module objects or comodule objects.

Working Definition 1.14 (Frobenius monads). The adjunctions appearing in this paper are *ambidextrous*: the left adjoint is canonically the right adjoint of its right adjoint and vice versa. Thus their compositions provide endofunctors with compatible monadic and comonadic structures. We refer to an endofunctor arising in this way as a *Frobenius monad*, though we do not independently formalize this notion in the ∞ -categorical setting (see [St, La] for the notion in the discrete setting). A natural context for considering Frobenius monads is the cobordism hypothesis with singularities [L3], where the notion of ambidextrous adjunction captures an oriented domain wall between topological field theories (as explained pictorially by [La] in the discrete setting).

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2. GROTHENDIECK-SPRINGER RESOLUTION

2.1. Recollections. We recall here the construction of the Grothendieck-Springer resolution of a reductive Lie algebra and the Steinberg variety.

Let G be a complex reductive group. For a Borel subgroup $B \subset G$, let $N \subset B$ denote its unipotent radical, and $H = B/N$ the universal Cartan torus. Denote by \mathfrak{g} , \mathfrak{b} , \mathfrak{n} , and \mathfrak{h} the respective Lie algebras. Let W denote the Weyl group of \mathfrak{g} , and $\mathfrak{c} = \mathfrak{h} // W$ the affine quotient. Fix a G -invariant inner product on \mathfrak{g} to obtain an identification $\mathfrak{g}^* \simeq \mathfrak{g}$.

Let $\mathcal{B} = G/B$ be the flag variety, and $\tilde{\mathcal{B}} = G/N$ the base affine space. The natural projection $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$ is a G -equivariant torsor for the natural H -action on $\tilde{\mathcal{B}}$. Such torsors correspond to homomorphisms $B \rightarrow H$, and the base affine space $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$ corresponds to the tautological homomorphism $B \rightarrow B/N \simeq H$.

²or more properly, *quasi-proper*

The cotangent bundle $T^*\mathcal{B} \rightarrow \mathcal{B}$ classifies pairs of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ together with an element $v \in (\mathfrak{g}/\mathfrak{b})^* \simeq \mathfrak{n}$. The moment map for the natural G -action is given by the projection

$$\mu_{\mathcal{B}} : T^*\mathcal{B} \longrightarrow \mathfrak{g}^* \simeq \mathfrak{g} \quad \mu_{\mathcal{B}}(\mathfrak{b}, v) = v$$

The cotangent bundle $T^*\tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}}$ classifies pairs of an element $x_{\mathfrak{b}} \in \tilde{\mathcal{B}}$ over a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ together with an element $v \in (\mathfrak{g}/\mathfrak{n})^* \simeq \mathfrak{b}$. The moment map for the natural $G \times H$ -action is given by the projection

$$\mu_{\tilde{\mathcal{B}}} : T^*\tilde{\mathcal{B}} \longrightarrow \mathfrak{g}^* \times \mathfrak{h}^* \simeq \mathfrak{g} \times \mathfrak{h} \quad \mu_{\tilde{\mathcal{B}}}(x_{\mathfrak{b}}, v) = (v, [v])$$

where $[v] \in \mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ denotes the image of $v \in \mathfrak{b}$.

The cotangent bundles are related by Hamiltonian reduction along the H -action

$$T^*\mathcal{B} = T^*(\tilde{\mathcal{B}}/H) \simeq (p_{\mathfrak{h}} \circ \mu_{\tilde{\mathcal{B}}})^{-1}(0)/H$$

where $p_{\mathfrak{h}} : \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{h}$ denotes projection.

We will be interested in the quotient $\tilde{\mathfrak{g}} = (T^*\tilde{\mathcal{B}})/H$ classifying a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ together with an element $v \in (\mathfrak{g}/\mathfrak{n})^* \simeq \mathfrak{b}$. The moment map for the G -action on $T^*\tilde{\mathcal{B}}$ descends to the Grothendieck-Springer resolution

$$\mu_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \quad \mu_{\tilde{\mathfrak{g}}}(\mathfrak{b}, v) = v$$

The Grothendieck-Springer resolution $\mu_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ is projective, generically finite and G -equivariant. Moreover its relative dualizing sheaf is canonically trivial (and hence the same is true of any base change of $\mu_{\tilde{\mathfrak{g}}}$). To see this last claim, recall we have fixed a G -invariant inner product on \mathfrak{g} to obtain a G -equivariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$. This in turn induces an isomorphism of lines $\wedge^{\dim \mathfrak{g}} \mathfrak{g} \simeq \wedge^{\dim \mathfrak{h}} \mathfrak{h}$. Thus a trivialization of $\wedge^{\dim \mathfrak{h}} \mathfrak{h}$ trivializes the canonical bundle of \mathfrak{g} . Furthermore, the partial moment map $\tilde{\mathfrak{g}} \rightarrow \mathfrak{h}$ is smooth with symplectic fibers, hence a trivialization of $\wedge^{\dim \mathfrak{h}} \mathfrak{h}$ also trivializes the canonical bundle of $\tilde{\mathfrak{g}}$.

2.1.1. The Grothendieck-Steinberg variety. The Grothendieck-Steinberg variety is the fiber product $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ classifying triples of a pair of Borel subalgebras $\mathfrak{b}_1, \mathfrak{b}_2 \subset \mathfrak{g}$ together with an element $v \in \mathfrak{b}_1 \cap \mathfrak{b}_2$. (Note here the derived fiber product coincides with the naive fiber product.)

It has a microlocal interpretation involving the double coset spaces

$$Z = B \backslash G / B \simeq G \backslash \mathcal{B} \times \mathcal{B} \quad \tilde{Z} = N \backslash G / N \simeq G \backslash \tilde{\mathcal{B}} \times \tilde{\mathcal{B}}.$$

Namely, returning to the identification

$$\tilde{\mathfrak{g}} = (T^*\tilde{\mathcal{B}})/H \quad \tilde{\mathcal{B}} = G/N$$

we have a similar identification

$$G \backslash (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \simeq (T^*\tilde{Z})/H \times H$$

or after de-equivariantization

$$\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \simeq (pt \times_{BG} T^*\tilde{Z})/H \times H.$$

From this viewpoint, the fiber product in the construction of $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ arises as the moment map equation for Hamiltonian reduction along the diagonal G -action for $T^*\tilde{\mathcal{B}} \times T^*\tilde{\mathcal{B}}$.

2.2. Descent pattern. Given a stack X , we will write $\mathcal{Q}(X)$ for the symmetric monoidal dg category of quasicoherent sheaves on X . All of the stacks X in play will be perfect in the sense of [BFN] and so the basic structure results for $\mathcal{Q}(X)$ will apply.

The natural G -action on $\tilde{\mathcal{B}} = G/N$ and induced Hamiltonian G -action on $\tilde{\mathfrak{g}}$ endows $\mathcal{Q}(\tilde{\mathfrak{g}})$ with two important compatible structures: an algebraic action of G as formalized by a $\mathcal{Q}(G)$ -module structure under convolution, and a $\mathcal{Q}(\mathfrak{g})$ -module structure via pullback under $\mu_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$. Altogether, these structures are captured by considering $\mathcal{Q}(\tilde{\mathfrak{g}}/G)$ as a $\mathcal{Q}(\mathfrak{g}/G)$ -module via the pullback under the induced map $\tilde{\mathfrak{g}}/G \rightarrow \mathfrak{g}/G$. We will identify the symmetries of $\mathcal{Q}(\tilde{\mathfrak{g}}/G)$ preserving this structure.

First, the endomorphisms of $\mathcal{Q}(\tilde{\mathfrak{g}})$ as a $\mathcal{Q}(\mathfrak{g})$ -module are given (thanks to [BFN]) by integral transforms with kernels on the fiber product: we have a monoidal equivalence

$$\Phi : \mathcal{Q}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \xrightarrow{\sim} \text{End}_{\mathcal{Q}(\mathfrak{g})}(\mathcal{Q}(\tilde{\mathfrak{g}})) \quad \Phi_{\mathcal{K}}(-) = p_{2*}(\mathcal{K} \otimes p_1^*(-))$$

where $p_1, p_2 : \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}$ denote the projections. In particular, the identity functor corresponds to the integral kernel $\Delta_{\tilde{\mathfrak{g}}}^* \mathcal{O}_{\tilde{\mathfrak{g}}} \in \mathcal{Q}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$ obtained by pushforward along the diagonal map

$$\Delta_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \longrightarrow \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$$

Similarly, the endomorphisms of $\mathcal{Q}(\tilde{\mathfrak{g}})$ as a $\mathcal{Q}(G)$ -module are given by integral transforms with G -equivariant kernels on the product

$$\text{End}_{\mathcal{Q}(G)}(\mathcal{Q}(\tilde{\mathfrak{g}})) \simeq \mathcal{Q}(G \backslash (\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}})).$$

Finally, the endomorphisms of $\mathcal{Q}(\tilde{\mathfrak{g}})$ as a Hamiltonian G -category, or in other words, the endomorphisms of $\mathcal{Q}(\tilde{\mathfrak{g}}/G)$ as a $\mathcal{Q}(\mathfrak{g}/G)$ -module, are given by integral transforms with equivariant kernels on the fiber product

$$\text{End}_{\mathcal{Q}(\mathfrak{g}/G)}(\mathcal{Q}(\tilde{\mathfrak{g}}/G)) \simeq \mathcal{Q}(G \backslash (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}))$$

or in other words, the monoidal dg category of equivariant quasicoherent sheaves on the Grothendieck-Steinberg variety.

2.2.1. Descent (co)monad. Consider the standard adjunction and Grothendieck duality adjunction on stable dg categories of quasi-coherent sheaves

$$\mu_{\tilde{\mathfrak{g}}}^* : \mathcal{Q}(\mathfrak{g}) \rightleftarrows \mathcal{Q}(\tilde{\mathfrak{g}}) : \mu_{\tilde{\mathfrak{g}}*} \quad \mu_{\tilde{\mathfrak{g}}*} : \mathcal{Q}(\tilde{\mathfrak{g}}) \rightleftarrows \mathcal{Q}(\mathfrak{g}) : \mu_{\tilde{\mathfrak{g}}}^!$$

Since the relative dualizing sheaf of $\mu_{\tilde{\mathfrak{g}}}$ is canonically trivial, we have a canonical equivalence $\mu_{\tilde{\mathfrak{g}}}^! \simeq \mu_{\tilde{\mathfrak{g}}}^*$, but we distinguish them to avoid confusion. By the projection formula, we can view these as adjunctions of $\mathcal{Q}(\mathfrak{g})$ -module categories. The adjunctions are also evidently G -equivariant, and in total preserve the Hamiltonian G -structure.

Let $T = \mu_{\tilde{\mathfrak{g}}}^! \mu_{\tilde{\mathfrak{g}}*}$ denote the resulting monad, or in other words, algebra object in the monoidal category of linear endomorphisms $\text{End}_{\mathcal{Q}(\mathfrak{g})}(\mathcal{Q}(\tilde{\mathfrak{g}}))$. Likewise, let $T^\vee = \mu_{\tilde{\mathfrak{g}}}^* \mu_{\tilde{\mathfrak{g}}*}$ denote the resulting comonad. Since $\mu_{\tilde{\mathfrak{g}}}^*$ is conservative and continuous, $\mu_{\tilde{\mathfrak{g}}}^!$ is as well; and since $\mu_{\tilde{\mathfrak{g}}}^!$ is cocontinuous, $\mu_{\tilde{\mathfrak{g}}}^*$ is as well. Thus the Barr-Beck theorem provides canonical identifications of Hamiltonian G -categories

$$\mathcal{Q}(\tilde{\mathfrak{g}})_T \simeq \mathcal{Q}(\mathfrak{g}) \simeq \mathcal{Q}(\tilde{\mathfrak{g}})^{T^\vee}$$

By base change and standard identities, the comonad T^\vee is given by tensoring with the sheaf of functions $\mathcal{O}_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}} = p_1^* \mathcal{O}_{\tilde{\mathfrak{g}}}$ with its canonical coalgebra structure. Likewise, the monad T is given by tensoring with the relative dualizing sheaf $\omega_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}} = p_1^! \mathcal{O}_{\tilde{\mathfrak{g}}}$ with its canonical algebra structure. Note that the identification of underlying functors $T \simeq T^\vee$ reflects the equivalence $p_1^* \simeq p_1^!$ which devolves by base change from the original ambidextrous adjunction of $\mu_{\tilde{\mathfrak{g}}}$.

2.3. Specified eigenvalues. We describe here the above descent picture to distinguished loci within \mathfrak{g} .

2.3.1. *Regular locus.* Over the open regular locus $\mathfrak{g}^r \subset \mathfrak{g}$, we have a fiber square

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}^r = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \mathfrak{g}^r & \xrightarrow{\mu_{\widetilde{\mathfrak{g}}^r}} & \mathfrak{g}^r \\ \downarrow & & \downarrow \\ \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{c} = \mathfrak{h}/W_{\mathfrak{g}} \end{array}$$

Thus descent over $\mathfrak{g}^r \subset \mathfrak{g}$ is simply the base change of descent over the geometric invariant theory quotient.

2.3.2. *Regular semisimple locus.* Over the open regular semisimple locus $\mathfrak{g}^{rs} \subset \mathfrak{g}$, we have a fiber square

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}^{rs} = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \mathfrak{g}^{rs} & \xrightarrow{\mu_{\widetilde{\mathfrak{g}}^{rs}}} & \mathfrak{g}^{rs} \\ \downarrow & & \downarrow \\ \mathfrak{h}^r & \xrightarrow{\pi^r} & \mathfrak{c}^r = \mathfrak{h}^r/W_{\mathfrak{g}} \end{array}$$

In other words, we have a free W -action and quotient identification

$$W_{\mathfrak{g}} \times \widetilde{\mathfrak{g}}^{rs} \longrightarrow \widetilde{\mathfrak{g}}^{rs} \quad \mathfrak{g}^{rs} \simeq \widetilde{\mathfrak{g}}^{rs}/W$$

Thus descent over $\mathfrak{g}^{rs} \subset \mathfrak{g}$ is simply equivariance for the Weyl group W .

2.3.3. *Nilpotent cone.* Over the nilpotent cone $\mathcal{N} = \mathfrak{g} \times_{\mathfrak{c}} \{0\} \subset \mathfrak{g}$, we have the base change

$$\mu_{\widetilde{\mathfrak{g}}_0} : \widetilde{\mathfrak{g}}_0 = \widetilde{\mathfrak{g}} \times_{\mathfrak{c}} \{0\} \longrightarrow \mathcal{N}$$

where $\widetilde{\mathfrak{g}}_0$ is a non-reduced scheme with underlying reduced scheme the usual Springer resolution $\widetilde{\mathcal{N}} \simeq T^*\mathcal{B}$ classifying a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ together with an element $v \in (\mathfrak{g}/\mathfrak{b})^* \simeq \mathfrak{n}$.

By construction, descent along $\mu_{\widetilde{\mathfrak{g}}_0}$ is governed by the restricted algebra object

$$\mathcal{O}_{\widetilde{\mathfrak{g}}_0 \times_{\mathcal{N}} \widetilde{\mathfrak{g}}_0} \simeq \mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}} |_{\mathcal{N}}$$

Remark 2.1. To work instead with the traditional reduced Springer resolution

$$\mu_{\widetilde{\mathcal{N}}} : \widetilde{\mathcal{N}} \longrightarrow \mathcal{N}$$

we must pass to ind-coherent sheaves. In applying the Barr-Beck theorem, we use that the adjunction

$$\mu_{\widetilde{\mathfrak{g}}_0*} : \mathcal{Q}(\widetilde{\mathfrak{g}}_0) \rightleftarrows \mathcal{Q}(\mathcal{N}) : \mu_{\widetilde{\mathfrak{g}}_0}^!$$

comprises a proper left adjoint and hence continuous right adjoint. But in contrast, this does not hold for the adjunction

$$\mu_{\widetilde{\mathcal{N}}*} : \mathcal{Q}(\widetilde{\mathcal{N}}) \rightleftarrows \mathcal{Q}(\mathcal{N}) : \mu_{\widetilde{\mathcal{N}}}^!$$

For example, $\widetilde{\mathcal{N}}$ is smooth, hence all skyscraper sheaves on it are compact, but \mathcal{N} is singular, hence many skyscraper sheaves on it are not compact. Rather we must pass to ind-coherent sheaves and work with the analogous adjunction

$$\mu_{\widetilde{\mathcal{N}}*} : \mathcal{Q}^!(\widetilde{\mathcal{N}}) \rightleftarrows \mathcal{Q}^!(\mathcal{N}) : \mu_{\widetilde{\mathcal{N}}}^!$$

Here by construction, the adjunction comprises a proper left adjoint and hence continuous right adjoint.

3. BEILINSON-BERNSTEIN LOCALIZATION

Now we will repeat the constructions of the previous section after quantization of the natural Poisson structures, that is, after turning on the noncommutative deformation from cotangent bundles to \mathcal{D} -modules.

3.1. Quantization. Let $\mathfrak{U}\mathfrak{g}$ be the universal enveloping algebra of \mathfrak{g} , and $\mathfrak{Z}\mathfrak{g} \subset \mathfrak{U}\mathfrak{g}$ the Harish Chandra center.

Let $\mathfrak{U}\mathfrak{g}\text{-mod}$ denote the dg category of $\mathfrak{U}\mathfrak{g}$ -modules. Informally speaking, $\mathfrak{U}\mathfrak{g}\text{-mod}$ consists of noncommutative modules on the Poisson manifold $\mathfrak{g} \simeq \mathfrak{g}^*$.

Let $\mathfrak{U}\mathfrak{g}\text{-perf} \subset \mathfrak{U}\mathfrak{g}\text{-mod}$ denote the small stable full dg subcategory of perfect modules so that $\mathfrak{U}\mathfrak{g}\text{-mod} \simeq \text{Ind}(\mathfrak{U}\mathfrak{g}\text{-perf})$.

Lemma 3.1. *There are canonical equivalences*

$$\mathfrak{U}\mathfrak{g}\text{-perf} \simeq \mathfrak{U}\mathfrak{g}\text{-perf}^{op} \quad \mathfrak{U}\mathfrak{g}\text{-mod} \simeq \mathfrak{U}\mathfrak{g}\text{-mod}^\vee$$

Proof. First, viewing $\mathfrak{U}\mathfrak{g}$ as a $\mathfrak{U}\mathfrak{g}$ -bimodule, we define the duality identification

$$\mathfrak{U}\mathfrak{g}\text{-perf}^{op} \xrightarrow{\sim} \mathfrak{U}\mathfrak{g}^{op}\text{-perf} \quad M \longmapsto \text{Hom}_{\mathfrak{U}\mathfrak{g}}(M, \mathfrak{U}\mathfrak{g}[\dim \mathfrak{g}])$$

Now let \mathfrak{g}^{op} denote the vector space \mathfrak{g} with the opposite Lie bracket

$$[\cdot, \cdot]_{\mathfrak{g}^{op}} = -[\cdot, \cdot]_{\mathfrak{g}}$$

The negation map $\mathfrak{g} \rightarrow \mathfrak{g}$, $v \mapsto -v$ provides a canonical isomorphism $\mathfrak{g} \simeq \mathfrak{g}^{op}$ and hence a canonical isomorphism $\mathfrak{U}\mathfrak{g} \simeq \mathfrak{U}\mathfrak{g}^{op}$. This establishes the first assertion, and the second then follows from the standard identity

$$\mathfrak{U}\mathfrak{g}\text{-mod}^\vee \simeq \text{Ind}(\mathfrak{U}\mathfrak{g}\text{-perf}^{op})$$

□

Remark 3.2. The equivalence $\mathfrak{U}\mathfrak{g}\text{-perf} \simeq \mathfrak{U}\mathfrak{g}\text{-perf}^{op}$ is a twisted form of the Serre duality equivalence $\text{Perf}(\mathfrak{g}^*) \simeq \text{Perf}(\mathfrak{g}^*)^{op}$. Namely, the former invokes the negation on the vector space \mathfrak{g}^* while the latter does not.

Let $\mathcal{D}_{\tilde{\mathcal{B}}} \in \mathcal{Q}(\tilde{\mathcal{B}})$ denote the sheaf of differential operators on $\tilde{\mathcal{B}}$. Let $\mathcal{D}_{mon}(\tilde{\mathcal{B}})$ denote the dg category of weakly H -equivariant \mathcal{D} -modules on $\tilde{\mathcal{B}}$. Its objects are H -equivariant quasicoherent sheaves on $\tilde{\mathcal{B}}$ equipped with a compatible H -equivariant action of $\mathcal{D}_{\tilde{\mathcal{B}}}$. Informally speaking, $\mathcal{D}_{mon}(\tilde{\mathcal{B}})$ consists of noncommutative modules on the Poisson manifold $\tilde{\mathfrak{g}} = (T^*\tilde{\mathcal{B}})/H$.

Let $\tilde{\mathcal{D}}_{\mathcal{B}} \in \mathcal{Q}(\mathcal{B})$ denote the sheaf of H -invariant differential operators on $\tilde{\mathcal{B}}$. It admits the natural presentation

$$\tilde{\mathcal{D}}_{\mathcal{B}} \simeq \mathcal{D}_{\mathcal{B}} \otimes_{\mathfrak{Z}\mathfrak{g}} \mathfrak{U}\mathfrak{h}$$

where $\mathfrak{U}\mathfrak{h}$ acts by vector fields on the right, and $\mathfrak{Z}\mathfrak{g} \subset \mathfrak{U}\mathfrak{g}$ by differential operators on the left. Then $\mathcal{D}_{mon}(\tilde{\mathcal{B}})$ is equivalently the dg category of quasicoherent sheaves on \mathcal{B} equipped with a compatible action of $\tilde{\mathcal{D}}_{\mathcal{B}}$.

Let $\mathcal{D}_{mon}^c(\tilde{\mathcal{B}}) \subset \mathcal{D}_{mon}(\tilde{\mathcal{B}})$ denote the small stable full ∞ -subcategory of coherent modules so that $\mathcal{D}_{mon}(\tilde{\mathcal{B}}) \simeq \text{Ind}(\mathcal{D}_{mon}^c(\tilde{\mathcal{B}}))$.

Lemma 3.3. *Verdier duality provides canonical equivalences*

$$\mathcal{D}_{mon}^c(\tilde{\mathcal{B}}) \simeq \mathcal{D}_{mon}^c(\tilde{\mathcal{B}})^{op} \quad \mathcal{D}_{mon}(\tilde{\mathcal{B}}) \simeq \mathcal{D}_{mon}(\tilde{\mathcal{B}})^\vee$$

Proof. The first assertion is Verdier duality, and the second follows from the standard identity $\mathcal{D}_{mon}(\tilde{\mathcal{B}})^\vee \simeq \text{Ind}(\mathcal{D}_{mon}^c(\tilde{\mathcal{B}})^{op})$. □

Remark 3.4. When keeping track of additional structures, it is useful to keep in mind the opposite base affine space $\tilde{\mathcal{B}}^{op} = N \backslash G$. The inverse map $G \rightarrow G$, $g \mapsto g^{-1}$ provides a canonical isomorphism $\tilde{\mathcal{B}} \simeq \tilde{\mathcal{B}}^{op}$.

Consider the localization adjunction

$$\begin{aligned} \gamma^* : \mathfrak{U}\mathfrak{g}\text{-mod} &\rightleftarrows \mathcal{D}_{mon}(\tilde{\mathcal{B}}) : \gamma_* \\ \gamma^*(M) &= \mathcal{D}_{\tilde{\mathcal{B}}} \otimes_{\mathfrak{U}\mathfrak{g}} M & \gamma_*(\mathcal{M}) &= \text{Hom}(\mathcal{D}_{\tilde{\mathcal{B}}}, \mathcal{M}) \end{aligned}$$

Informally speaking, this is a quantization of the standard adjunction for the Grothendieck-Springer resolution $\mu_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$.

Proposition 3.5. *The right adjoint γ_* is continuous and proper, and hence itself admits a continuous right adjoint $\gamma^!$. Furthermore, there is a canonical identification $\gamma^! \simeq \gamma^*$.*

Proof. It suffices to check that the right adjoint γ_* fits into a commutative square on compact objects

$$\begin{array}{ccc} \mathcal{D}_{mon}^c(\tilde{\mathcal{B}}) & \simeq & \mathcal{D}_{mon}^c(\tilde{\mathcal{B}})^{op} \\ \gamma_* \downarrow & & \downarrow \gamma_*^{op} \\ \mathfrak{U}\mathfrak{g}\text{-perf} & \simeq & \mathfrak{U}\mathfrak{g}\text{-perf}^{op} \end{array}$$

But this is evident from its construction and that of the horizontal duality equivalences. \square

3.1.1. *Linearity.* Let $\mathfrak{Z}\mathfrak{g} \subset \mathfrak{U}\mathfrak{g}$ be the Harish Chandra center, and $\mathfrak{U}\mathfrak{h}$ the universal enveloping algebra of \mathfrak{h} . We also have the canonical embedding $\mathfrak{Z}\mathfrak{g} \subset \mathfrak{U}\mathfrak{h}$ as the ρ -shifted Weyl invariants.

Observe that $\mathfrak{U}\mathfrak{g}\text{-mod}$ is naturally $\mathfrak{Z}\mathfrak{g}$ -linear, and $\mathcal{D}_{mon}(\tilde{\mathcal{B}})$ is naturally $\mathfrak{U}\mathfrak{h}$ -linear and hence $\mathfrak{Z}\mathfrak{g}$ -linear. The following is evident from the constructions.

Lemma 3.6. *The adjunctions*

$$\gamma^* : \mathfrak{U}\mathfrak{g}\text{-mod} \rightleftarrows \mathcal{D}_{mon}(\tilde{\mathcal{B}}) : \gamma_* \quad \gamma_* : \mathcal{D}_{mon}(\tilde{\mathcal{B}}) \rightleftarrows \mathfrak{U}\mathfrak{g}\text{-mod} : \gamma^!$$

are naturally $\mathfrak{Z}\mathfrak{g}$ -linear.

3.1.2. *Symmetries.* We next introduce quantum analogues of the previously encountered Hamiltonian G -actions. Following [BD, FG], by a *smooth G -category*, we mean a dg category with an algebraic G -action that is infinitesimally trivialized, or in other words, the induced action of the formal group of G is trivialized. This can be formalized by saying that a dg category is a module for the monoidal dg category $\mathcal{D}(G)$ of \mathcal{D} -modules on G under convolution.

The primary examples are the dg category $\mathcal{D}(X)$ of \mathcal{D} -modules on a G -variety X , and the dg category $\mathfrak{U}\mathfrak{g}\text{-mod}$ with its conjugation G -action. These can be unified by considering more generally $\mathcal{D}(G)$ -modules of the form $\mathcal{D}_{G'\text{-mon}}(X)$ where X is a $G \times G'$ -variety, and we take weakly G' -equivariant \mathcal{D} -modules. In particular, for $G = X = G'$ with G acting on the left and G' on the right, we have the $\mathcal{D}(G)$ -linear equivalence $\mathfrak{U}\mathfrak{g}\text{-mod} \simeq \mathcal{D}_{G'\text{-mon}}(G)$. Informally speaking, this is a quantum analogue of the identification $\mathfrak{g}^* \simeq (T^*G)/G$. Similarly, $\mathcal{D}_{mon}(\tilde{\mathcal{B}})$ comes equipped with a natural $\mathcal{D}(G)$ -module structure.

Remark 3.7. Within the stable setting, there are two equivalent dual formulations of a smooth G -category. By definition, we have taken smooth G -category to mean a module for the monoidal dg category $\mathcal{D}(G)$ of \mathcal{D} -modules on G under convolution. But one can observe that for X a smooth variety, $\mathcal{D}(X)$ is dualizable as a plain dg category. Furthermore, it is self-dual so that for maps $f : X \rightarrow Y$ of smooth varieties, pullback is dual to pushforward (by the projection formula). Thus a smooth G -category could equivalently be taken to mean a comodule for $\mathcal{D}(G)$ equipped with its coconvolution coalgebra structure.

The following is evident from the constructions.

Lemma 3.8. *The adjunctions*

$$\gamma^* : \mathfrak{U}\mathfrak{g}\text{-mod} \rightleftarrows \mathcal{D}_{mon}(\tilde{\mathcal{B}}) : \gamma_* \quad \gamma_* : \mathcal{D}_{mon}(\tilde{\mathcal{B}}) \rightleftarrows \mathfrak{U}\mathfrak{g}\text{-mod} : \gamma^!$$

are naturally $\mathcal{D}(G)$ -linear.

Consider the stack $\tilde{Z} = N \backslash G / N$. Let $\mathcal{D}_{bimon}(\tilde{Z})$ denote the dg category of $H \times H$ -weakly equivariant \mathcal{D} -modules on \tilde{Z} . Informally speaking, $\mathcal{D}_{bimon}(\tilde{Z})$ consists of noncommutative modules on the Grothendieck-Steinberg stack $G \backslash (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$.

Convolution equips $\mathcal{D}_{bimon}(\tilde{Z})$ with a natural monoidal structure, and $\mathcal{D}_{mon}(\tilde{\mathcal{B}})$ with a natural right $\mathcal{D}_{bimon}(\tilde{Z})$ -module structure commuting with its natural left $\mathcal{D}(G)$ -module structure. This is a quantum analogue of the convolution pattern for sheaves on $\tilde{\mathfrak{g}} \times_{\tilde{\mathfrak{g}}} \tilde{\mathfrak{g}}$ acting by integral transforms on sheaves on $\tilde{\mathfrak{g}}$ respecting the Hamiltonian G -structure. The following is the quantum analogue of the result quoted from [BFN] earlier that such integral transforms are precisely the symmetries respecting the Hamiltonian G -structure.

Theorem 3.9. *Convolution provides a monoidal equivalence*

$$\Phi : \mathcal{D}_{bimon}(\tilde{Z}) \xrightarrow{\sim} \text{End}_{\mathcal{D}(G)}(\mathcal{D}_{mon}(\tilde{\mathcal{B}}))$$

Proof. Let us begin by forgetting the $\mathcal{D}(G)$ -module structure of $\mathcal{D}_{mon}(\tilde{\mathcal{B}})$. Then by [BN], we have a monoidal equivalence

$$\Phi' : \mathcal{D}_{bimon}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \xrightarrow{\sim} \text{End}(\mathcal{D}_{mon}(\tilde{\mathcal{B}})) \quad \Phi'(\mathcal{K})(-) = p_{2*}(\mathcal{K} \otimes p_1^*(-))$$

where $p_1, p_2 : \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}}$ denote the projections.

Returning $\mathcal{D}(G)$ -module structures to the picture, Φ' is evidently $\mathcal{D}(G)$ -linear by standard identities. Moreover, $\mathcal{D}(G)$ -linear endomorphisms of $\mathcal{D}_{mon}(\tilde{\mathcal{B}})$ are simply the invariants

$$\text{End}_{\mathcal{D}(G)}(\mathcal{D}_{mon}(\tilde{\mathcal{B}})) = \text{Hom}_{\mathcal{D}(G)}(\mathcal{D}(pt), \text{End}(\mathcal{D}_{mon}(\tilde{\mathcal{B}})))$$

By descent along $pt \rightarrow BG$, the invariants can be calculated as comodules

$$\text{Hom}_{\mathcal{D}(G)}(\mathcal{D}(pt), \text{End}(\mathcal{D}_{mon}(\tilde{\mathcal{B}}))) \simeq \text{End}(\mathcal{D}_{mon}(\tilde{\mathcal{B}}))^{\mathcal{O}_G}$$

for the canonical coalgebra $\mathcal{O}_G \in \mathcal{D}(G)$ given by the structure sheaf.

On the other hand, by another application of descent, \mathcal{O}_G -comodules in $\mathcal{D}_{bimon}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$ are precisely G -equivariant bimonodromic \mathcal{D} -modules on $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$. Now the theorem follows from the identification $\tilde{Z} \simeq G \backslash (\tilde{\mathcal{B}} \times \tilde{\mathcal{B}})$. \square

3.2. Universal Weyl sheaf. Recall the adunctions

$$\gamma^* : \mathfrak{U}\mathfrak{g}\text{-mod} \xrightleftharpoons{\sim} \mathcal{D}_{mon}(\tilde{\mathcal{B}}) : \gamma_* \quad \gamma_* : \mathcal{D}_{mon}(\tilde{\mathcal{B}}) \xrightleftharpoons{\sim} \mathfrak{U}\mathfrak{g}\text{-mod} : \gamma^!$$

with ambidextrous identification $\gamma^! \simeq \gamma^*$.

Let $T = \gamma^! \gamma_*$ denote the resulting monad, or in other words, algebra object in the monoidal category of endomorphisms $\text{End}_{\mathcal{D}(G)}(\mathcal{D}_{mon}(\tilde{\mathcal{B}}))$. Likewise, let $T^\vee = \gamma^* \gamma_*$ denote the resulting comonad.

Lemma 3.10. *$\gamma^!$ is conservative, and hence γ^* is as well.*

Proof. From the standard calculation

$$\gamma_*(\mathcal{D}_{\tilde{\mathcal{B}}}) = \text{Hom}(\mathcal{D}_{\tilde{\mathcal{B}}}, \mathcal{D}_{\tilde{\mathcal{B}}})^H \simeq \mathfrak{U}\mathfrak{g} \otimes_{\mathfrak{Z}\mathfrak{g}} \mathfrak{U}\mathfrak{h}$$

we find

$$\text{Hom}(\mathcal{D}_{\tilde{\mathcal{B}}}, \gamma^!(M)) \simeq \text{Hom}(\mathfrak{U}\mathfrak{g} \otimes_{\mathfrak{Z}\mathfrak{g}} \mathfrak{U}\mathfrak{h}, M)$$

Factoring with the canonical integration morphism $\mathfrak{U}\mathfrak{g} \otimes_{\mathfrak{Z}\mathfrak{g}} \mathfrak{U}\mathfrak{h} \rightarrow \mathfrak{U}\mathfrak{g}$, we conclude that $M \neq 0$ implies $\gamma^!(M) \neq 0$. \square

Since $\gamma^!$ is continuous and conservative and γ^* is cocontinuous and conservative, the Barr-Beck theorem provides canonical identifications

$$\text{Mod}_T(\mathcal{D}_{mon}(\tilde{\mathcal{B}})) \simeq \mathfrak{U}\mathfrak{g}\text{-mod} \simeq \text{Comod}_{T^\vee}(\mathcal{D}_{mon}(\tilde{\mathcal{B}}))$$

We would like to explicitly describe the integral kernel giving rise to $T \simeq T^\vee$ under the equivalence

$$\Phi : \mathcal{D}_{bimon}(\tilde{Z}) \xrightarrow{\sim} \text{End}_{\mathcal{D}(G)}(\mathcal{D}_{mon}(\tilde{\mathcal{B}}))$$

Definition 3.11. The *universal Weyl sheaf* $\mathcal{W} \in \mathcal{D}_{bimon}(\tilde{Z})$ is the sheaf of differential operators on \tilde{Z} with its canonical $H \times H$ -weakly equivariant structure.

Remark 3.12. By quantum Hamiltonian reduction (and under the identification $\tilde{Z} \simeq G \backslash (G/N \times G/N)$), the pullback of \mathcal{W} along the natural quotient map

$$r : G/N \times G/N \longrightarrow N \backslash G/N$$

is the G -strongly equivariant \mathcal{D} -module

$$r^* \mathcal{W} = \mathcal{D}_{G/N \times G/N} / (\mathfrak{g})$$

where $(\mathfrak{g}) \subset \mathcal{D}_{G/N \times G/N}$ is the left ideal generated by vector fields arising from the diagonal G -action on $G/N \times G/N$. Thus \mathcal{W} is the quantum analogue of the structure sheaf of $G \backslash (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$.

Remark 3.13. One can write down an explicit Frobenius algebra structure on \mathcal{W} but its construction is an explicit unwinding of that given by the adjunctions of Theorem 3.14 below.

In parallel with the commutative case, the identification of \mathcal{W} as the integral kernel giving rise to $T \simeq T^\vee$ can be viewed as a microlocal version of G -equivariant base change along the diagram

$$\begin{array}{ccc} & \mathcal{D}_{H \times H}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) & \\ \swarrow p_{1*} & & \nwarrow p_2^* \\ \mathcal{D}_{mon}(\tilde{\mathcal{B}}) & & \mathcal{D}_{mon}(\tilde{\mathcal{B}}) \\ \swarrow \gamma^* & & \searrow \gamma_* \\ & \mathfrak{U}\mathfrak{g}\text{-mod} & \end{array}$$

This is formalized in the argument for the following assertion.

Theorem 3.14. *The monoidal equivalence*

$$\Phi : \mathcal{D}_{bimon}(\tilde{Z}) \xrightarrow{\sim} \text{End}_{\mathcal{D}(G)}(\mathcal{D}_{mon}(\tilde{\mathcal{B}}))$$

takes the universal Weyl sheaf \mathcal{W} to the endofunctor $T \simeq T^\vee$. Thus \mathcal{W} inherits the structures of algebra and coalgebra in $\mathcal{D}_{bimon}(\tilde{Z})$, and we have equivalences

$$\mathcal{D}_{mon}(\tilde{\mathcal{B}})_{\mathcal{W}} \simeq \mathfrak{U}\mathfrak{g}\text{-mod} \simeq \mathcal{D}_{mon}(\tilde{\mathcal{B}})^{\mathcal{W}}.$$

Proof. On the one hand, given an object $\mathcal{M} \in \mathcal{D}_{mon}(\tilde{\mathcal{B}})$, we find

$$T(\mathcal{M}) \simeq \gamma^* \gamma_* \mathcal{M} \simeq \tilde{\mathcal{D}}_{\mathcal{B}} \otimes_{\mathfrak{U}\mathfrak{g}} \text{Hom}_{\tilde{\mathcal{D}}_{\mathcal{B}}}(\tilde{\mathcal{D}}_{\mathcal{B}}, \mathcal{M})$$

Thus it is the $\mathfrak{U}\mathfrak{g}$ -coinvariants of the intermediate functor

$$T'(\mathcal{M}) = \tilde{\mathcal{D}}_{\mathcal{B}} \otimes \text{Hom}_{\tilde{\mathcal{D}}_{\mathcal{B}}}(\tilde{\mathcal{D}}_{\mathcal{B}}, \mathcal{M})$$

Moreover, under the identification

$$\Phi' : \mathcal{D}_{bimon}(\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}) \xrightarrow{\sim} \text{End}(\mathcal{D}_{mon}(\tilde{\mathcal{B}}))$$

observe that T' corresponds to the integral kernel $\tilde{\mathcal{D}}_{\mathcal{B}} \boxtimes \tilde{\mathcal{D}}_{\mathcal{B}}$.

On the other hand, as discussed above, \mathcal{W} is simply the $\mathfrak{U}\mathfrak{g}$ -coinvariants of $\tilde{\mathcal{D}}_{\mathcal{B}} \boxtimes \tilde{\mathcal{D}}_{\mathcal{B}}$. Thus since all functors are continuous, taking $\mathfrak{U}\mathfrak{g}$ -coinvariants can be equivalently performed on the integral kernel or on the result of the integral transform.

All of the above arguments are manifestly G -equivariant and so descend to give the assertion. \square

3.3. Specified infinitesimal character. Recall that the commutative algebra $\mathfrak{U}\mathfrak{h} \otimes \mathfrak{U}\mathfrak{h} = \mathcal{O}(\mathfrak{h}^* \times \mathfrak{h}^*)$ acts by central endomorphisms on $\mathcal{D}_{bimon}(\tilde{Z})$. The action factors through the closed subscheme $\Gamma \subset \mathfrak{h}^* \times \mathfrak{h}^*$ given by the union of the graphs of Weyl elements

$$\Gamma = \coprod_{w \in W} \Gamma_w \quad \Gamma_w = \{(\lambda, w\lambda) \in \mathfrak{h}^* \times \mathfrak{h}^*\}$$

To better understand the universal Weyl sheaf $\mathcal{W} \in \mathcal{D}_{bimon}(\tilde{Z})$, let us restrict one of its monodromies and calculate its resulting fiber. The composite projection to either factor

$$\Gamma \hookrightarrow \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$$

is a finite flat map. All of what follows is symmetric in the two projections, so let us focus on the projection to the first factor.

First, let us identify the fibers of $\mathcal{D}_{bimon}(\tilde{Z})$ along the projection to the first factor. For simplicity, let us also forget the H -weak equivariance along the second factor. For any $\lambda \in \mathfrak{h}^*$, the corresponding fiber of $\mathcal{D}_{bimon}(\tilde{Z})$ is canonically equivalent to the dg category $\mathcal{D}_\lambda(N \setminus \mathcal{B})$ of N -strongly equivariant λ -twisted \mathcal{D} -modules on the flag variety \mathcal{B} . This in turn is the full subcategory of those λ -twisted \mathcal{D} -modules on \mathcal{B} that are locally constant along Schubert cells.

Now let us identify the corresponding fiber

$$\mathcal{W}_\lambda \in \mathcal{D}_\lambda(N \setminus \mathcal{B})$$

of the universal Weyl sheaf. This is a regular holonomic λ -twisted \mathcal{D} -module on the flag variety \mathcal{B} locally constant along Schubert cells.

For concreteness, we will consider several specific cases: (1) λ generic (regular and not at all integral), (2) λ regular and integral, and (3) $\lambda = 0$ trivial.

(1) When λ is generic, we have a direct sum decomposition

$$\mathcal{W}_\lambda \simeq \bigoplus_{w \in W} \mathcal{W}_{\lambda, w\lambda}$$

Each summand admits the description as a standard or equivalently costandard extension off of a Schubert cell

$$\mathcal{W}_{\lambda, w\lambda} \simeq j_{w!} \mathcal{O}_{\lambda, w} \simeq j_{w*} \mathcal{O}_{\lambda, w}$$

where $j_w : \mathcal{B}_w \rightarrow \mathcal{B}$ is the inclusion of the w -Schubert cell for $w \in W$, and $\mathcal{O}_{\lambda, w}$ is the λ -twisted structure sheaf of \mathcal{B}_w .

(2) Suppose λ is regular and integral, and let $w_0 \in W$ be the Weyl element such that $\lambda = w_0 \lambda_0$ for dominant λ_0 .

We can tensor by the line bundle $\mathcal{O}(\lambda)$ to obtain an identification

$$\mathcal{D}(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}_\lambda(\mathcal{B}) \quad M \longmapsto M \otimes \mathcal{O}(\lambda)$$

This is convenient since we will describe \mathcal{W}_λ in terms of the natural monoidal structure on the dg category $\mathcal{D}(B \setminus \mathcal{B})$ of B -equivariant \mathcal{D} -modules on the flag variety \mathcal{B} . Namely, under the above identification, we have a direct sum decomposition

$$\mathcal{W}_\lambda \simeq \bigoplus_{w \in W} \mathcal{W}_{\lambda, w\lambda}$$

Each summand admits the description as the convolution of standard extensions $T_{w*} = j_{w*} \mathcal{O}_w$ and costandard extensions $T_{w!} = j_{w!} \mathcal{O}_w$ off of Schubert cells

$$\mathcal{W}_{\lambda, w\lambda} \simeq T_{w_0!} * T_{ww_0^{-1}*}$$

(3) When $\lambda = 0$ is trivial, the fiber \mathcal{W}_0 is the maximal tilting sheaf on \mathcal{B} . Namely, within $\mathcal{D}(N \setminus \mathcal{B})$, it is the projective cover of the skyscraper sheaf at the closed Schubert cell.

3.4. Example: SL_2 . We offer here a brief discussion of the structure of the universal Weyl sheaf \mathcal{W} in the case when $G = SL_2$. Already here one can see how intricate topology is packaged in the simple algebra of \mathcal{W} . More specifically, the simple notion of differential operators on $N \backslash SL_2 / N$ interpolates between standard, costandard and tilting sheaves as we specialize parameters.

Let us identify $\mathfrak{h}^* \simeq \mathbb{A}^1$ so that $\Lambda^* \simeq \mathbb{Z}$ with the reflection action of the Weyl group $W \simeq \mathbb{Z}/2$ centered at $-\rho = -1 \in \mathbb{Z} \subset \mathbb{A}^1$. Thus $-1 \in \mathbb{Z} \subset \mathbb{A}^1$ is the unique singular parameter, the rest of the integers $\mathbb{Z} \setminus \{-1\}$ are regular integral, and the rest of the parameters $\mathfrak{h}^* \setminus \mathbb{Z}$ are generic (not at all integral).

Let us identify

$$SL_2/N \simeq \mathbb{A}^2 \setminus \{(0,0)\} = \text{Spec } \mathbb{C}[x,y] \setminus \{(0,0)\}$$

so that the left action of B has orbits

$$i : V = \mathbb{G}_m \times \{0\} \hookrightarrow SL_2/N$$

and the complement

$$j : U = \mathbb{A}^1 \times \mathbb{G}_m \hookrightarrow SL_2/N$$

and the right action of $H \simeq \mathbb{G}_m$ is the usual scaling dilation. Thus the left action of the diagonal torus $T \subset B$ coincides with the right action of H on the closed orbit V and is its inverse on the open orbit U . Note that the left action of $N \simeq \mathbb{A}^1$ consists of the individual points of the closed orbit V and the slices $\mathbb{A}^1 \times \{y\} \subset U$ of the open orbit.

The N -equivariant ring of differential operators on G/N is given by Hamiltonian reduction

$$\mathcal{W} = \mathcal{D}_{G/N}/(\mathfrak{n})$$

where $(\mathfrak{n}) \subset \mathcal{D}_{G/N}$ is the left ideal generated by the vector field $y\partial_x$ arising from the left N -action. Thus the coisotropic (but not Lagrangian) singular support of \mathcal{W} is the union of conormals to N -orbits and explicitly cut out by the single equation $y\xi_x = 0$.

Let us specialize \mathcal{W} to prescribed monodromies for the right H -action. For $\lambda \in \mathfrak{h}^* \simeq \mathbb{A}^1$, the corresponding fiber is the quotient

$$\mathcal{W}_\lambda = \mathcal{D}_{G/N}/(\mathfrak{n}, \mathfrak{h}_\lambda)$$

where $(\mathfrak{n}, \mathfrak{h}_\lambda) \subset \mathcal{D}_{G/N}$ is the left ideal generated by the vector field $y\partial_x$ arising from the left N -action, and the differential operator $x\partial_x + y\partial_y - \lambda$ coming from prescribing the monodromy of the right H -action. Thus \mathcal{W}_λ is regular holonomic with singular support the union of conormals to B -orbits and explicitly cut out by the equations $y\xi_x = x\xi_x + y\xi_y = 0$. From these equations, we see its characteristic cycle is the weighted sum of conormals to B -orbits

$$cc(\mathcal{W}_\lambda) = 2 \cdot T_V^* + T_U^*$$

Now when $\lambda \in \mathfrak{h}^* \setminus \mathbb{Z}$ is generic, \mathcal{W}_λ splits as a direct sum

$$\mathcal{W}_\lambda = i_*\mathcal{L}_{V,\lambda} \oplus j_*\mathcal{L}_{U,\lambda}$$

of the standard extensions of the local systems on B -orbits

$$\mathcal{L}_{V,\lambda} = \mathcal{D}_V/(x\partial_x - \lambda) \quad \mathcal{L}_{U,\lambda} = \mathcal{D}_U/(\partial_x, y\partial_y - \lambda)$$

Observe that because λ is generic, the standard extension off of the open orbit U is equivalent to the costandard extension

$$j_!\mathcal{L}_{U,\lambda} \simeq j_*\mathcal{L}_{U,\lambda}$$

(Of course, there is no difference in the standard and costandard extensions off of the closed orbit V .) The characteristic cycles of the summands are the sums of conormals to B -orbits

$$cc(i_*\mathcal{L}_{V,\lambda}) = T_V^* \quad cc(j_*\mathcal{L}_{U,\lambda}) = T_V^* + T_U^*$$

When $\lambda \in \mathbb{Z} \setminus \{-1\}$ becomes regular integral, \mathcal{W}_λ continues to split as a direct sum but now in a more delicate form. Namely, when $\lambda \in \{0, 1, 2, \dots\}$ is “positive”, we have

$$\mathcal{W}_\lambda = i_*\mathcal{L}_{V,\lambda} \oplus j_*\mathcal{L}_{U,\lambda}$$

and when $\lambda \in \{-2, -3, -4, \dots\}$ is “negative”, we have

$$\mathcal{W}_\lambda = i_* \mathcal{L}_{V, \lambda} \oplus j_* \mathcal{L}_{U, \lambda}$$

From the perspective of the monadic symmetries governing localization, this asymmetry reflects the choice of global sections functor. For example, going back to Borel-Weil-Bott, we see that the global sections of line bundles leads to the asymmetry of cohomological shifts.

Finally, when $\lambda = -1$ is singular, \mathcal{W}_0 no longer splits as a direct sum but becomes the indecomposable tilting extension of the structure sheaf of the open orbit

$$\mathcal{W}_0 = \mathcal{O}_U^{tilt} = \mathcal{D}_{G/N} / (y\partial_x, x\partial_x + y\partial_y)$$

Thus it is self-dual and admits an increasing filtration

$$\mathcal{W}_0^0 \subset \mathcal{W}_0^1 \subset \mathcal{W}_0$$

with associated graded

$$\mathcal{W}_0^0 \simeq i_* \mathcal{O}_V \quad \mathcal{W}_0^1 / \mathcal{W}_0^0 \simeq \mathcal{O}_{G/N} \quad \mathcal{W}_0 / \mathcal{W}_0^1 \simeq i_* \mathcal{O}_V$$

To see any of the preceding identifications of \mathcal{W}_λ explicitly, one can restrict to the transverse line $\mathbb{A}^1 \simeq \{x = 1\}$ to find

$$\mathcal{W}_\lambda|_{\{x=1\}} \simeq \mathcal{D}_{\mathbb{A}^1} / (y^2\partial_y - \lambda y) \simeq \mathcal{D}_{\mathbb{A}^1} / (y\partial_y y - (\lambda + 1)y)$$

In particular, when $\lambda = -1$ is singular, we find the traditional algebraic presentation $\mathcal{D}_{\mathbb{A}^1} / (y\partial_y y)$ of the tilting extension of the structure sheaf of \mathbb{G}_m to all of \mathbb{A}^1 .

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